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AN ALTERNATIVE INTERPRETATION OF THE PRIMAL-DUAL METHOD AND SOME RELATED PARAMETRIC METHODS

by

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Abstract

The primal-dual method is interpreted as a parametric linear programming method. Some variants and related methods, such as Dantzig's self-dual parametric method and the linear programming variant of Houthakker's capacity method for quadratic programming are considered and compared.

1. INTRODUCTION

In a well-known paper by Dantzig, Ford, and Fulkerson (2), a method for solving a linear programming problem was presented. In effect, they gave a set of rules which they proved would lead ultimately to an optimal solution - if one existed. In the course of presenting the specific rules of the method, they gave an interpretation to the algorithm involving the formulation of both the primal and the dual problems. Here we give an alternative interpretation of the method leading to the same set of rules.

It turns out that the primal-dual method can be interpreted as a parametric method of a very simple sort. In the following, this method is explained by means of a simple example. The equivalence with the primal-dual method is then considered in some detail.

The parametric problem concerned is one with a parametric objective function. Its dual problem has a parametric right-hand side. This problem may be solved parametrically. Any problem which has an initial feasible solution can be treated in the same manner; the resulting method may be called the dual equivalent of the primal-dual method. This method is closely related to the linear programming variant of Houthakker's capacity method in its simplicial formulation. The latter method can be proved to be equivalent to Dantzig's self-dual parametric method. These matters are discussed in the last section.

2. A PARAMETRIC METHOD FOR LINEAR PROGRAMMING

We consider the following linear programming problem. Minimize

(2.1) $$f = c'x$$

subject to

(2.2) $$Ax = b$$

(2.3) $$x \geq 0$$

c and x are column vectors of n elements, b is a column vector of m elements and A is an m x n matrix; the symbol ' denotes transposition. We assume that the elements of b and c are nonnegative.[1] We also assume that there is at least one feasible

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[1] For cases in which the elements of c are not nonnegative, see Dantzig, Ford and Fulkerson (2) or Dantzig (1). There it is proposed to add an "artificial constraint" $e'x \leq \theta$ for a large unspecified value of θ and to generate a feasible solution to the dual problem.

solution to the constraints (2.2), (2.3).

Let us also consider the following related problem which we shall call the extended problem. Minimize

(2.4) $$f^* = c'x + \lambda e'y$$

subject to

(2.5) $$Ax + y = b,$$

(2.6) $$x, y \geq 0.$$

y is a column vector of m artificial variables, e is a vector of m elements which are all unity and λ is a variable parameter.

For a sufficiently high value of λ, the solution of the extended problem must be the same as that of the original problem, because such a value of λ will prevent the y-variables from having nonzero values. On this, the usual two-phase method for linear programming is based. The extended problem with λ having a very high value is solved instead of the original problem with an initial basic solution $y = b$. This amounts to minimizing first the objective function

(2.7) $$e'y = \Sigma y,$$

since the terms in λ are dominant; this leads to a feasible solution of the original problem, after which the original objective function can be used to find the optimal solution.

The following parametric method also uses the extended problem, but instead of solving the problem immediately for a high value of λ, we solve the problem first for $\lambda = 0$, after which λ is increased parametrically. The initial basic feasible solution $y = b$ is $\lambda = 0$ also an optimal solution, since all elements of c are assumed to be nonnegative. After that, parametric linear programming (see Gass and Saaty (3))is used to trace the optimal solutions of the extended problem for increasing values of λ. The solution for $\lambda \to \infty$ must be the optimal solution of the original problem.

This parametric method is equivalent to the primal-dual method in the sense that its computational rules are the same; it can therefore be viewed as an alternate interpretation of the primal-dual method. First an example of application of this parametric method will be given. In the next section, this example will be used to explain the equivalence.

As an example, we take the problem used by Dantzig (1) for the primal-dual method. Minimize

(2.8)
$$f = x_1 + 4x_2 + 8x_3 + 8x_4 + 23x_5$$

subject to

(2.9)
$$\begin{cases} x_1 + 4x_2 - 5x_3 + 7x_4 - 4x_5 = 8, \\ \quad - 4x_2 + 4x_3 - 4x_4 + 4x_5 = 2, \\ \qquad x_2 - 3x_3 + 4x_4 - 2x_5 = 2, \end{cases}$$

(2.10)
$$x_1,\ x_2,\ x_3,\ x_4,\ x_4 \geq 0.$$

In the formulation of the extended problem, the terms

$$\lambda y_1 + \lambda y_2 + \lambda y_3$$

are added to the objective function and y_1, y_2 and y_3 are added to the left side of the respective equations of (2.9). The initial basic solution is then $y_1 = 8$, $y_2 = 2$ and $y_3 = 2$ and the corresponding initial tableau is obtained by subtracting λ times the equality constraints of the extended problem from its objective function. This objective function becomes then

(2.11)
$$f^* = x_1 + 4x_2 + 8x_3 + 8x_4 + 23x_5 + \lambda(-12 - x_1 - x_2 + 4x_3 - 7x_4 + 2x_5).$$

Putting the λ terms in a separate row, the initial tableau as given in Tableau 0 of Table 1 is obtained; the value of the terms without λ is indicated by f, that of the terms with λ by w. Adding to the f-row λ times, the w-row for specific values of λ, a row is obtained which represents the objective function for specific values of λ.

In Tableau 0 of Table 1, the specific value of λ is first taken to be 0. For this value, the initial solution is also an optimal solution of the extended problem, since all coefficients in the row $-f^*(0)$ are non-negative. Next, consider for what range of λ the present solution is an optimal one. Its upper bound is determined by

$$\min_{i} \left(\frac{f_i}{-w_i} \;\middle|\; w_i < 0 \right), \tag{2.12}$$

where f_i stands for the element in the f-row and in the i-th column and w_i for the element in the w-row and in the same column. In Tableau 0, it turns out that the highest value of λ for which the solution is optimal is 1, because for that value the coefficient of x_1 in the objective function, becomes zero. The row $-f^*(1)$ gives then the value of the objective function for $\lambda = 1$. According to the usual parametric procedure, x_1 enters the basis and y_1 leaves it. Tableau 0 is then transformed into Tableau 1. Note that the value of w has decreased, as it should, because a variable having a negative coefficient in the w-row entered the basis. The column of y_1 which is now a nonbasic variable is deleted because we do not wish y_1 to re-enter the basis.

The solution of Tableau 1 is optimal for $\lambda = 1$. An upper bound on λ for which this solution is optimal is found by applying (2.12) again; this upper bound turns out to be $\lambda = 13$.

TABLE 1. SIMPLEX TABLEAUX FOR EXAMPLE

Tabl.	Basic Var.	Values Bas. V.	Nonbasic Variables				
			x_1	x_2	x_3	x_4	x_5
0	y_1	8	1	4	-5	7	-4
	y_2	2	0	-4	4	-4	4
	y_3	2	0	1	-3	4	-2
	-w	-12	-1	-1	4	-7	2
	-f	0	1	4	8	8	23
	-f*(0)	0	1	4	8	8	23
	-f*(1)	-12	0	3	12	1	25
				x_2	x_3	x_4	x_5
1	x_1	8		4	5	7	-4
	y_2	2		-4	4	-4	4
	y_3	2		1	-3	4	-2
	-w	-4		3	-1	0	-2
	-f	-8		0	13	1	27
	-f*(1)	-12		3	12	1	25
	-f*(13)	-60		39	0	1	1
				x_2		x_4	x_5
2	x_1	10½		-1		2	1
	x_3	½		-1		-1	1
	y_3	3½		-2		1	1
	-w	-3½		2		-1	-1
	-f	-14½		13		14	14
	-f*(13)	-60		39		1	1
	-f*(14)	-63½		41		0	0
				x_2			x_5
3	x_1	3½		3			-1
	x_3	4		-3			2
	x_4	3½		-2			1
	-w	0		0			0
	-f	-63½		41			0
	-f*(14)	-63½		41			0
				x_2			x_3
4	x_1	5½		1½			½
	x_5	2		-1½			½
	x_4	1½		-½			-½
	-f	-63½		41			0

x_3 must then enter the basis and y_2 leaves it; the column of y_2 is deleted in the resulting tableau.

In Tableau 2, it turns out that the minimum in (2.12) is not unique, because the coefficients of x_4 and x_5 in the f*-row both vanish for $\lambda = 14$. This is a degenerate solution, but it causes no difficulty because whichever variable enters into the basis, the value of w decreases. Hence, either variable may enter the basis. Choosing x_4, we find that y_3 must leave the basis. The solution of the resulting tableau is found to be the optimal solution, since w has a zero value. This optimal solution is not unique because the coefficient of f in the column of x_5 is zero. The corresponding extreme-point optimal solution is generated in Tableau 4.

3. EQUIVALENCE WITH THE PRIMAL-DUAL METHOD

The primal-dual method starts with a feasible solution to the dual of the original problem. The general form of this dual problem is, see (2.1)-(2.3):

Maximize

(3.1) $$b'u$$

subject to

(3.2) $$A'u \leqq c.$$

u is a column vector of m elements. Introducing a vector v of n slack variables, the constraint (3.2) can be written as

(3.3) $$A'u + v \leqq c,$$

(3.4) $$v \geqq 0.$$

Because it was assumed that the elements of c are nonnegative, an initial feasible solution is $v = c$, $u = 0$. However, this solution is not likely to be an optimal one, since all elements of b are nonnegative.

We consider also the primal feasibility problem: Minimize

(3.5) $$e'y$$

subject to

(3.6) $$Ax + y = b,$$

(3.7) $$x, y \geqq 0,$$

and its dual: Maximize

(3.8) $$b'u^*$$

subject to

(3.9) $$A'u^* \leqq 0,$$

(3.10) $$u^* \leqq e.$$

Constraint (3.9) can be written as

(3.11) $$A'u^* + v^* = 0,$$

(3.12) $$v^* \geqq 0.$$

The primal-dual method is based on the following ideas. An initial solution of the dual problem (3.1)-(3.2) is available. Suppose there are some vectors u^*, v^* which give a positive value of the objective function of the dual feasibility problem; for this latter solution, (3.12) need not be satisfied. If the first solution is $\bar{u}$, $\bar{v}$, and the second $\bar{u}^*$, $\bar{v}^*$, then the solution

(3.13) $$\bar{u} + k\bar{u}^*, \ \bar{v} + k\bar{v}^*$$

must give a higher value of the objective function of the dual problem for $k > 0$, since $b'\bar{u}^*$ was assumed to be positive. However, $\bar{v}^*$ was not necessarily positive, so that for some value of $k \geqq 0$, $\bar{v} + k\bar{v}^*$ might become negative. Hence we determine

(3.14) $$\bar{k} = \operatorname*{Min}_{i} \left(\frac{\bar{v}_i}{-\bar{v}_i^*} \;\middle|\; \bar{v}_i^* < 0 \right);$$

If this value of k is used in (3.13), the objective function of the dual is increased as much as possible without making its solution infeasible. After this, a new solution to the dual feasibility problem is generated and added to the solution of the dual in the same manner as before. The dual objective function is increased until no improvement is possible because the objective function of the dual feasibility problem has become zero. The optimal solution of the dual problem has then been obtained.

The solutions of the dual feasibility problem are obtained via the primal feasibility problem; the dual variables appear then in the row of the objective function. The method starts usually with a solution of the dual problem $v = c$, $u = 0$, which is a feasible solution. For the primal feasibility problem, the solution $y = b$, $x = 0$ is taken. The corresponding basic solution of the dual feasibility problem is

$$u^* = e, \quad v^* = -A'u^* = -A'e; \tag{3.15}$$

the value of its objective function is $b'e$ which is positive if b has at least one positive element. Adding a multiple k determined by (3.14) of the solution (3.15) to that of the dual problem, we find that the objective function of the dual problem is increased for $k \neq 0$; $k = 0$ can only occur if some basic v-variables are zero for corresponding negative v^*-variables. This can only occur in the first iteration, because there are some additional requirements for solutions of the feasibility problem in later iterations. These are, given the improved solution of the dual problem, a <u>restricted</u> primal infeasibility problem is solved, in which the variables to enter the basis are restricted to those which have zero corresponding variables in the dual problem. These will be the basic variables and the variable connected with k in (3.14); if the minimum was not unique, then all the variables connected with this minimum are included plus possible other variables connected with zero v-variables. The optimal solution to this restricted primal feasibility problem is one with v*-variables of the corresponding dual which are nonnegative for v-variables which are zero. Hence when we next add the solution of the corresponding dual feasibility problem to the dual solution, k is nonzero. After this, another restricted primal feasibility problem is solved and so on, until the objective function of the feasibility problem has become zero; in that case the optimal solution of the dual problem has been found, and also the solution of the original problem.

Let us now compare the primal-dual method with the parametric approach using the numerical example presented in the previous section. Tableau 0 of Table 1 gives the initial solution of the primal feasibility problem; the w-row gives the value of its objective function and the values of the basic variables of the corresponding dual solution. Hence we have

$$v_1^* = -1, \quad v_2^* = -1, \quad v_3^* = 4, \quad v_4^* = -7, \quad v_5 = 2.$$

The f-row now gives the values of the objective function of the original dual problem and its corresponding solution. The solution of the dual problem is

$$v_1 = 1, \quad v_2 = 4, \quad v_3 = 8, \quad v_4 = 8, \quad v_5 = 23.$$

Now k times the w-row is added to the f-row, thus increasing the objective function of the dual from 0 to 12k. The maximum value of k turns out to be 1; the row $-f^*(1)$ gives the improved value of the objective function of the dual as well as the corresponding solution of the dual. According to the parametric procedure, x_1 must enter the basis. The same is true for the primal-dual method, since in the restricted primal feasibility problem only x_1 and the basic variables may be in the basis. In the resulting transformation both methods transform the rows of basic variables and the w-row in the same way; the primal-dual method has no f-row and it does not transform the present solution of the dual problem. The parametric procedure transforms the f-row or $-f^*(1)$-row, but this last row does not change since its element in the x_1-column is zero.

In the next tableau the primal-dual method adds k times the w-row to the $f^*(1)$. $\bar{k}$ is then found to be 12 and is connected with x_3. The parametric procedure adds λ times the w-rows to the f-row and finds $\lambda = 12$, connected with x_3. The result, the row indicated by $-f^*(13)$ is, in both cases, the same. As is easily seen, k is equal to the increase in λ.

Each cycle in the primal-dual method corresponds with a particular value of λ in the parametric approach. The restricted columns of the primal feasibility problem are the same as the columns which have the same ratio λ of elements in the f- and the w-row. Usually the optimal solution to the restricted primal feasibility problem will be obtained in one iteration, but it is possible that it takes more iterations. This can be the case when the maximum from which k or λ is found is not unique. The adjusted dual solution contains then more than one zero apart from the basic variables. An example of this can be found in Tableau 2, where both x_4 and x_5 are connected with the minimum in (3.14). x_4 and x_5 are then both columns of the restricted primal feasibility problem and if x_5 is chosen as a basic variable instead of x_4, it takes two iterations to obtain the solution of the restricted problem. Again there is no substantial difference with the parametric procedure.

Hence it may be concluded that both procedures are equivalent and differ only by having k in the primal-dual method which is equal to the increment of λ used in the parametric method. The two methods may therefore be seen as two alternative interpretations of the same algorithm.

4. VARIANTS AND RELATED PARAMETRIC METHODS

The dual of the extended problem is as follows. Maximise

$$f = b'u \tag{4.1}$$

subject to

$$A'u \leqq c, \tag{4.2}$$

$$u \leqq \lambda e. \tag{4.3}$$

The last set of constraints does not occur in the original problem but corresponds to the artificial variables in the extended problem.

Each method for the primal linear programming problem has its equivalent method for the dual problem; for instance, the dual method for the primal problem is equivalent to the Simplex method for the dual problem. We shall now indicate the equivalent method of the parametric method described in Section 2.

The initial tableau can be obtained as follows. The constraints (4.2) and (4.3) can be written as equations by means of the introduction of the vectors of slack variables v and z:

(4.4) $$A'u + v = c,$$

(4.5) $$u + z = \lambda e,$$

(4.6) $$v, z \geq 0.$$

Putting the equations (4.1), (4.4) and (4.5) into tableau format, we obtain the set-up tableau given in Table 2; the values of basic variables are separated into a constant term and a term dependent on λ. In the equivalent primal method the y-variables were basic and the x-variables nonbasic in the initial solution. In the dual equivalent the z-variables must therefore be nonbasic and the v-variables basic; the u-variables must be basic because there are no slack variables in the primal problem. This initial solution and its corresponding tableaux are generated by block-pivoting on the underlined matrix I in the set-up tableau in order to introduce the u-variables in the basis, replacing the z-variables. After the u-variables have entered the basis, their rows may be deleted, because the u-variables are unrestricted.

TABLE 2. SET-UP AND INITIAL TABLEAU FOR EQUIVALENT DUAL METHOD

	Bas. Var.	Val.B.Var. C.t	Val.B.Var. λ-t	u	v	z	f
Set-Up Tabl.	v	c	0	A'	I	0	0
	z	0	e	<u>I</u>	0	I	0
	f	0	0	-b'	0	0	1
Init. Tabl.	v	c	-A'e	0	I	-A'	0
	u	0	e	I	0	I	0
	f	0	b'e	0	0	b'	1

The initial solution is optimal for $\lambda = 0$ since b and c are both nonnegative. After this, λ is increased parametrically until $\lambda \to \infty$. This method is completely equivalent to the primal-dual method.

The dual equivalent of the primal-dual method may be applied to any linear programming problem with an initial feasible solution. Consider the following problem. Maximize

$$f = c'x \tag{4.7}$$

subject to

$$Ax \leqq b, \tag{4.8}$$

$$x \geqq 0. \tag{4.9}$$

We assume that both b and c are nonnegative. Adding the "artificial constraint"

$$x \leqq \lambda e, \tag{4.10}$$

we find that the problem has the same form as (4.1) - (4.3) apart from the fact that the x-variables are restricted. The same method may now be applied. Table 3 gives the set-up and initial tableaux of this method which may be called the dual equivalent of the primal-dual method.

Let us consider an application of this method to the following small problem. Maximize

$$f = 3x_1 + 4x_2 \tag{4.11}$$

subject to

$$-x_1 + 2x_2 \leqq 2, \tag{4.12}$$

$$x_1 - x_2 \leqq 1, \tag{4.13}$$

$$x_1, x_2 \geqq 0. \tag{4.14}$$

TABLE 3. SET-UP AND INITIAL TABLEAU FOR THE DUAL EQUIVALENT OF THE PRIMAL-DUAL METHOD

	Bas. Var.	Val. B. Var.		x	y	z	f
		C.t	λ-t.				
Set-Up Tableau	y	b	0	A	I	0	0
	z	0	e	I	0	I	0
	f	0	0	-c'	0	0	1
Init. Tableau	y	b	-Ae	0	I	-A	0
	x	0	e	I	0	I	0
	f	0	c'e	0	0	c'	1

Table 4 gives the successive solutions of the method. Tableau 0 is constructed in accordance with the initial tableau of Table 3. The critical value of λ is found to be 1; at this value y_1 has to leave the basis. Dual feasibility requires then that z_2 should enter the basis. The next critical value of λ is 4; hence y_2 should leave the basis and it is found that z_1 enters it. In Tableau 2 no critical value of λ can be found, so that the optimal solution must have been found. Not all rows given in Table 4 are needed; the rows of the basic z-variables can be deleted; furthermore, the rows of basic x-variables with corresponding nonbasic z-variables are always the same so that they do not have to be written down.

Instead of adding an "artificial constraint" for each variable as in (4.10), we may just add one "artificial constraint" on the sum of the variables,

$$e'x \leqq \lambda. \tag{4.15}$$

An initial feasible and optimal solution for $\lambda = 0$ is then found by introducing into the basis in the set-up tableau based on (4.7), (4.8) and (4.15), the variable with the largest c-coefficient, replacing the slack variable of (4.15). After this, λ is varied parametrically until $\lambda \to \infty$. Table 5 gives an application of this method to the problem (4.11) - (4.14). This procedure can be considered as a variant of Houthakker's capacity method for quadratic programming (4). For details, see van de Panne and Whinston (5).

TABLE 4. APPLICATION OF THE DUAL EQUIVALENT
OF THE PRIMAL-DUAL METHOD

Tableau	Basic Variable	Values Basic Variables			
		C.t	λ - t	z_1	z_2
0	y_1	2	-1	1	-2
	y_2	1	0	-1	1
	x_1	0	1	1	0
	x_2	0	1	0	1
	f	0	7	3	4
				z_1	y_1
1	z_2	-1	½	-½	-½
	y_2	2	-½	-½	½
	x_1	0	1	1	0
	x_2	1	½	½	½
	f	4	5	5	2
				y_2	y_1
2	z_2	-3	1	-1	-1
	z_1	-4	1	-2	-1
	x_1	4	0	2	1
	x_2	3	0	1	1
	f	24	0	10	7

TABLE 5. APPLICATION OF THE CAPACITY METHOD
TO A LINEAR PROGRAMMING PROBLEM

Tabl.	Bas.V.	Val.Bas. Var.		x_1	x_2	Tabl.	Bas.V.	Ind.t.	λ-t	y_1	y_3
		C.t.	λ-t.								
0	y_1	2	0	-1	2	2	x_1	-2/3	2/3	-1/3	2/3
	y_2	1	0	1	-1		y_2	2 1/3	-1/3	2/3	$\underline{-1/3}$
	y_3	0	1	1	$\underline{1}$		x_2	2/3	1/3	1/3	1/3
	f	0	0	-3	-4		f	2/3	$3\frac{1}{3}$	1/3	$3\frac{1}{3}$
				x_1	y_3					y_1	y_2
1	y_1	2	-2	$\underline{-3}$	-2	3	x_1	4	0	1	2
	y_2	1	1	2	1		y_3	-7	1	-2	-3
	x_2	0	1	1	1		x_2	3	0	1	1
	f	0	4	1	4		f	24	0	7	10

A method which is related to the above approach is Dantzig self-dual parametric method.[2] First we shall deal with the special case in which the problem has the form (4.7) - (4.9) with b nonnegative, and the elements of c positive. The following objective function is then used:

(4.16) $$f = (c - \lambda e)'x.$$

For

(4.17) $$\lambda \geq \max_j c_j,$$

the initial solution in which the slack variables of (4.8) are basic is feasible and optimal. A parametric procedure is then used to decrease λ until an optimal solution is found for λ = 0 this is then the solution of the original problem.

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[2] See Dantzig (1) pp. 245 - 247.

TABLE 6. APPLICATION OF DANTZIG'S

SELF-DUAL PARAMETRIC METHOD

Tabl.	Bas.V.	V.B.V.	x_1	x_2
0	y_1	2	-1	2
	y_2	1	1	-1
	$f(\lambda)$	0	1	1
	$f(c)$	0	-3	-4
			x_1	y_1
1	x_2	1	-½	½
	y_2	2	½	½
	$f(\lambda)$	-1	1½	-½
	$f(c)$	4	-5	2
			y_2	y_1
2	x_2	3	1	1
	x_1	4	2	1
	$f(\lambda)$	-7	-3	-2
	$f(c)$	24	10	7

Table 6 gives the iterations for an application of this method to the problem (4.11) - (4.14). The first critical value of λ turns out to be 4, the second 3 $^1/3$; after the second iteration the optimal solution has been found.

For problems with all elements of b nonnegative and those of c positive the self-dual parametric method is equivalent to the capacity method for linear programming in the sense that it generates the same sequence of solutions which, however, differ in some respects. In the capacity method the slack variable of the capacity constraint will be nonbasic in all tableaux except the first and last ones, whereas it may be considered as a basic variable in the parametric method. A further difference is that the variable which in a certain iteration leaves the basis in the self-dual parametric iteration, leaves the basis in the capacity method in the corresponding next iteration; y_1 leaves the basis in the first iteration in the capacity method (see Tableau 0 of Table 6), while in the capacity method it leaves the basis in the second iteration (see Tableau 1 of Table 5).

The equivalence of both methods can be proved by induction as follows. In the first iteration the same variable enters the basis in both methods. In the capacity method y_λ, the slack variable of the capacity constraint, leaves the basis, while in the parametric method an ordinary basic variable leaves the basis.

Consider any tableau generated by the parametric method. A representation of such a tableau is given in the first tableau of Table 7; x_i stands for a typical basic variable, x_k for a typical nonbasic variable and x_j for the variable which left the basis in the last iteration. The corresponding tableau for the capacity method can be generated from this tableau by introducing x_j into the basis, replacing f_λ as a basic variable. The resulting tableau is then the second one of Table 7. We shall now check whether the variables to leave and to enter the basis are the same in both methods. In the capacity method the variable to leave the basis is determined by

$$(4.18) \qquad \min_i \left(\frac{b_i - a_{ij} d_j^{-1} d_0}{a_{ij} d_j^{-1}} , \frac{d_j^{-1} d_0}{-d_j^{-1}} \;\middle|\; -a_{ij} d_j^{-1} < 0 \right) .$$

TABLE 7. EQUIVALENCE OF SELF-DUAL PARAMETRIC AND CAPACITY METHOD

	Bas. Var.	Val. B. V.		f_λ	.	x_j	.	x_k	.
	.	.			.	.	.	.	.
Self-Dual Par. Meth.	x_i	b_i		0	.	a_{ik}	.	a_{ik}	
	.	.		.	.	.	.	.	.
	f_λ	λ_0		1	.	$\underline{d_j}$	.	d_k	
	f_c	c_0		0	.	c_j	.	c_k	.
	Bas. Var.	Val. Bas. Var. C.t.	Val. Bas. Var. λ-t.		.	x_j	.	x_k	.
	.	.	.		.	.	.	.	.
Cap. Meth.	x_i	$b_i - a_{ij} d_j^{-1} d_0$	$-a_{ij} d_j^{-1}$		.	0	.	$a_{ik} - a_{ij} d_j^{-1} d_k$	.
	.	.	.		.	.	.	.	.
	x_j	$d_j^{-1} d_0$	d_j^{-1}		.	1	.	$d_j^{-1} d_k$	.
	f_c	$c_0 - c_j d_j^{-1} d_0$	$-c_j d_j^{-1}$		.	0	.	$c_k - c_j d_j^{-1} d_k$	.

Since $d_j < 0$ because x_j left the basis, this may be rewritten as

$$(4.19) \qquad \min_i \left(-d_0 + \frac{b_i d_j}{a_{ij}} , -d_0 \;\middle|\; a_{ij} < 0 \right)$$

It is then obvious that the term behind the comma is connected with the minimum so that x_j leaves the basis, which is as it should be.

Also the variable to enter the basis is the same. In the parametric method it is determined by

(4.20) $$\min_k \left(\frac{c_k}{d_k} \;\middle|\; d_k > 0 \right),$$

while in the capacity method it is given by

(4.21) $$\min_k \left(\frac{c_k - c_j d_j^{-1} d_k}{-d_j^{-1} d_k} \;\middle|\; d_j^{-1} d_k < 0 \right),$$

which can be rewritten as

(4.22) $$\min_k \left(-d_j \frac{c_k}{d_k} + c_j \;\middle|\; d_k > 0 \right);$$

for $d_j < 0$ this is the same as (4.20).

In his self-dual parametric method Dantzig adds terms in λ only to x-variables which have positive coefficients in the objective function, while Houthakker includes all x-variables in this capacity constraint. This difference is a rather superficial one, since in the parametric method terms in λ may be added to the coefficients of all x-variables regardless of sign, while on the other hand, in the capacity method x-variables may be deleted from the capacity constraint if they have nonpositive coefficients in the objective function.

The main feature of the self-dual parametric method is, of course, that if a solution is neither primally nor dually feasible, terms in λ may also be added to the values of basic variables in order to make this solution feasible; hence if all elements of b are negative, b is replaced by

(4.23) $$b + \lambda e;$$

in case only some elements of b are negative we have instead of e a vector with unit elements corresponding with negative elements of b and zeros elsewhere. A parametric decrease in λ affects then also primal feasibility and the method may involve primal as well as dual iterations.

A corresponding capacity method can be constructed as follows. Consider the following problem. Maximize

(4.24) $$f = c'x - \lambda x_\lambda$$

subject to

(4.25) $$Ax - ex_\lambda + y = b,$$

(4.26) $$e'x + y_\lambda = \lambda,$$

(4.27) $$x,\ x_\lambda,\ y,\ y_\lambda \geqq 0.$$

Instead of the e-vectors we may again have vectors with unit elements only for negative elements of b and positive ones of c. x_λ is here an artificial variable corresponding with a dual capacity constraint

(4.28) $$e'u \leqq \lambda.$$

In the first iteration the x-variable corresponding with the largest positive element of c is introduced into the basis, replacing y_λ; in the second one, x_λ enters the basis, replacing the y-variable corresponding with the largest (in absolute value) negative element of b. After this, λ is increased parametrically in the usual fashion.

Comparing this version of the capacity method with the self-dual parametric method, it is found that the latter method is simpler, because its tableaux have one column and one row less, while furthermore the capacity method requires no extra iterations. The self-dual parametric method should therefore be preferred to the capacity method.

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13 ABSTRACT

The primal-dual method is interpreted as a parametric linear programming method. Some variants and related methods, such as Dantzig's self-dual parametric method and the linear programming variant of Houthakker's capacity method for quadratic programming are considered and compared.

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